Universal rules for suboptimal balls packing or minimized correlation vector frames

Equiangular tight frame (ETF) is a frame of vectors $\boldsymbol{M} = [\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots \boldsymbol{m}_N] \in \mathbb{E}^{D \times N}$, $(D < N,\ \mathbb{E} = \mathbb{R}\ or\ \mathbb{C})$, which satisfies following conditions (see Fig. 1):

- $\|\boldsymbol{m}_n\|_2 = 1,\ \forall n = 1, \ \ldots N$
- $|\langle \boldsymbol{m}_n, \boldsymbol{m}_k \rangle| = \mu\ \forall n \neq k$ for some $\mu > 0$
- $\boldsymbol{M}\boldsymbol{M}^H = \frac{N}{D}\boldsymbol{I}$

The minimal possible value of dot products for complex case can be shown to be $\mu = \sqrt{\frac{N-D}{D(N-1)}}$ - **Welch bound** value known previously [1, 2], where D is the length of vector (dimension of the space) and N is the number of vectors in frame, $N > D > \sqrt{N}$.

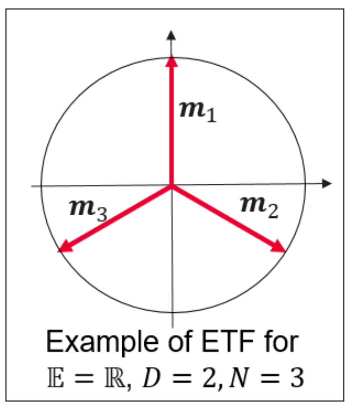

Fig. 1 *Simple Example of ETF, also known as Mercedes frame*

ETFs are very important in various domains, such as vector approximation, compressed sensing and multiple access.

Search for an ETFs is a complicated problem. There are pairs (N, D) for which ETFs are proven to exist. Among them there are some pairs (N, D) for which the construction of ETF is known. Moreover, there are some pairs (N, D) for which it is proven that ETFs do not exist [3].

There are number of methods to construct ETF and one of them is based on difference set concept [4]. D rows with difference set indexes of Fourier matrix with size N form ETF, problem of existence for such kind of ETF is the problem of existence for difference set with given N, D.

The main drawback is that the number of pairs N, D for which difference set construction is known is very limited. It is interesting to search for constructions that do not reach Welsh Bound precisely, but are available for larger number of N, D pairs.

The ETF problem for D-dimensional complex vectors can also be formulated as problem of packing open balls in a complex projective space $\mathbb{P}^{D-1}(\mathbb{C})$ [5] equipped with a distance

$$dist(\boldsymbol{x}, \boldsymbol{y}) = \arccos\left(\frac{|\langle \boldsymbol{x}, \boldsymbol{y}\rangle|}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2}\right)$$

The optimization task can be formulated as follows:

$$[\boldsymbol{m}_1,\ \boldsymbol{m}_2,\ \ \dots \boldsymbol{m}_N] = \arg \max_{\widehat{\boldsymbol{m}}_1,\ \widehat{\boldsymbol{m}}_2,\ \dots \widehat{\boldsymbol{m}}_N \neq 0} \min_{i \neq j} dist(\boldsymbol{m}_i, \boldsymbol{m}_j)$$

In particular, we are interested in DFT-like constructions:

$$\boldsymbol{m}_i[j] = \frac{1}{\sqrt{D}} \exp\left(i \frac{2\pi}{N} d_j\ i\right), j = 1 \dots D, i = 1, \dots N$$

In such case we only need to optimize coefficients d_j.

Another important requirement is that the final solution should not be an iterative algorithm to be executed for every pair (N, D). The final solution should rely on some universal design rule valid for various pairs (N, D).

We would greatly appreciate any insights, recommendations, or innovative approaches that could assist in addressing this challenge. Your expertise and contributions could significantly enhance our collaborative efforts, paving the way for deeper exploration and fruitful future partnerships.

References

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